Critical Multitype Branching Systems: Extinction Results

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Abstract

We consider a critical branching particle system in $\mathbb{R}^d$, composed of individuals of a finite number of types $i \in \{1, \ldots, K\}$. Each individual of type i moves independently according to a symmetric α_i-stable motion. We assume that the particle lifetimes and offspring distributions are type-dependent. Under the usual independence assumptions in branching systems, we prove extinction theorems in the following cases: (1) all the particle lifetimes have finite mean, or (2) there is a type whose lifetime distribution has heavy tail, and the other lifetimes have finite mean. We get a more complex dynamics by assuming in case (2) that the most mobile particle type corresponds to a finite-mean lifetime: in this case, local extinction of the population is determined by an interaction of the parameters (offspring variability, mobility, longevity) of the long-living type and those of the most mobile type. The proofs are based on a precise analysis of the occupation times of a related Markov renewal process, which is of independent interest.

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1 Introduction

In critical branching and migrating populations, mobility of individuals counteracts the tendency to asymptotic local extinction caused by the clumping effect of the branching. In fact, convergence to a non-trivial equilibrium may occur in a spatially distributed population whose members perform migration and reproduction, even if the branching is critical, provided that the mobility of individuals is strong enough. This behavior has been investigated in several branching models, including branching random walks [7, 9], Markov branching systems (both with monotype [6] and multitype [3, 4, 8] branching), and age-dependent branching systems [15].

In [15] Vatutin and Wakolbinger investigated a monotype branching model in Euclidean space $\mathbb{R}^d$, in which each particle moves according to a symmetric α-stable motion, and at

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the end of its lifetime it leaves at its death site a random number of offsprings, with critical offspring generating function $f(s) = s + \frac{1}{2}(1-s)^{1+\beta}$, $\beta \in (0,1]$. It turned out that, if the initial population is Poisson with uniform intensity and the particle lifetime distribution has finite mean, such process suffers local extinction if $d \le \alpha/\beta$, while for $d > \alpha/\beta$ the system is persistent, i.e. preserves its intensity in the large time limit. This result is consistent with the intuitive meaning of the population parameters: the exponent $\alpha > 0$ is the mobility parameter of individuals in the sense that a smaller α means a more mobile migration (i.e. more spreading out of particles) which is clearly in favor of persistence; β is the offspring variability parameter, meaning that a smaller β causes a stronger clustering effect in the population, which favors local extinction due to criticality of the branching. If the lifetime distribution has a power tail $t^{-\gamma}$ for some $\gamma \in (0,1]$, then the critical dimension is $\alpha\gamma/\beta$. Again, it is intuitively clear that long lifetimes (i.e. small γ) enhance the spreading out of individuals. However, Vatutin and Wakolbinger discovered that, in contrast with the case of finite-mean lifetimes, if the lifetimes have a general distribution of the above sort, the "critical" dimension does not necessarily pertains to the local extinction regime: when $d = \alpha\gamma/\beta$ persistence of the population is not excluded.

Our aim in the present paper is to get a better understanding about how population characteristics such as mobility, offspring variability, and longevity of individuals determine the asymptotic local extinction of branching populations. In order to attain this we deal with a multitype system, where the most mobile migration (corresponding to the smallest α) and the life-time distribution with the heaviest tail, may correspond to different particle types. More precisely, we consider a branching population living in $\mathbb{R}^d$, constituted of particles of different types $i \in \mathbf{K} := \{1, \ldots, K\}$. Each particle of type i moves according to a symmetric α_i-stable motion until the end of its random lifetime, which has a non-arithmetic distribution function Γ_i. Then it branches according to a multitype offspring distribution with generating function $f_i(\mathbf{s})$, $\mathbf{s} \in [0,1]^K$, $i \in \mathbf{K}$. The descendants appear where the parent individual died, and evolve independently in the same manner. The movements, lifetimes and branchings of particles are assumed to be independent; the only dependency in the system is that the offsprings start where the parent particle died. In addition, we assume that the process starts off at time 0 from a Poisson random population, with a prescribed intensity measure, and that all particles at time 0 have age 0. Let $M = (m_{i,j})_{i,j=1}^K$ denote the mean matrix of the multitype branching law, that is

$$m_{i,j} = \frac{\partial f_i}{\partial x_j}(\mathbf{1}),$$

where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^K$. We assume that $\mathbf{f}(\mathbf{s}) = (f_1(\mathbf{s}), \ldots, f_K(\mathbf{s})) \ne M\mathbf{s}$, and that M is an ergodic stochastic matrix. This implies that the branching is critical, i.e. the largest eigenvalue of M is 1.

For the system described above, here we investigate parameter configurations under which the population becomes locally extinct in the large time run. We deal first with the case when all particle lifetimes have finite mean and prove that the process suffers local extinction if $d < \alpha/\beta$, where the mobility parameter $\alpha = \min_{1 \le i \le K} \alpha_i$ is the same as in the Markovian case [8], and the offspring variability parameter $\beta \in (0,1]$ is determined by

$$x - \langle \mathbf{v}, \mathbf{1} - \mathbf{f}(\mathbf{1} - \mathbf{u}\,x)\rangle \sim x^{1+\beta} L(x) \text{ as } x \to 0,$$

where $\mathbf{v}$ denotes the (normalized) left eigenvector of the matrix M corresponding to the eigenvalue 1, and L is slowly varying at 0 in the sense that $\lim_{x\to 0} L(\lambda x)/L(x) = 1$ for every $\lambda > 0$. In a way, this case is similar to the one with exponentially distributed lifetimes.

Next we assume that exactly one particle type is long-living, i.e. its lifetime distribution has a power tail decay $t^{-\gamma}$, $\gamma \in (0,1]$, while the other lifetime types have distributions with tails decaying not slower than $A\,t^{-\eta}$ for some $\eta > 1$, $A > 0$. We consider two scenarios. In the first one we assume that the most mobile particle type is, at the same time, long-living, and we prove that extinction holds when $d < \alpha\gamma/\beta$. Then we proceed with the most interesting case: the most mobile particle type corresponds to a finite-mean lifetime. In this scenario, it turns out that local extinction of the population is determined by a complex interaction of the parameters (offspring variability, mobility, longevity) of the long-living type and those of the most mobile type. Assuming without loss of generality that type 1 is the long-living type, we prove that the systems suffers local extinction provided that $d < d_+$, where

$$d_+ = \frac{\gamma}{\frac{(\beta+1)\gamma}{\alpha} - \frac{1}{\alpha_1}}.$$

The positive number $\gamma\alpha_1$ can be considered as the "effective mobility" parameter of the long-living type. If $\gamma\alpha_1$ is very close to α (so that γ/α and $1/\alpha_1$ are approximately the same), then d_+ is also close to $\alpha_1\gamma/\beta$ and to α/β. Moreover, for fixed α, α_1 and γ, the parameter d_+ considered as a function of β, is decreasing, which is consistent with previous known results.

The proofs of our results rely on a precise asymptotic analysis for the occupation times in the branching particle system. Some of our techniques combine parts of the approaches used in [3] and [15] adapted to our model, however the adaptation to our case is far from being straightforward. In Section 2 we provide a family tree analysis which allows us to compare the occupation times of the particle system with the occupation times of an auxiliary Markov renewal process.

Then, in Section 3, we carry out the asymptotic analysis mentioned above by investigating the occupation times of all types in the auxiliary renewal process, as well as the asymptotic number of renewals in large time-intervals. This is the mathematical core of the paper, and we think it is interesting on its own right. To achieve this, we need to control the tail decay of the renewal times of all types simultaneously, which we were able to do assuming that there is only one long-living particle type. Therefore, in its present form our approach is not yet applicable to treat a general model with arbitrary lifetime distributions.

Finally, in Section 4 we give the extinction results in our various different setups. Let us remark that, when the particle lifetimes have finite mean and the spatial dimension is small, local extinction of the population can be proved without the occupation times analysis; in this case a simple estimation yields the result, see the proof of Theorem 1. In contrast, the occupation time analysis is needed to treat the case of long-living particle types.

2 Family tree analysis

Following [3] (p. 553–558) we introduce the following auxiliary process. Consider a Markov renewal process with values in $\mathbf{K}$, where in type i the process spends time according to a

non-lattice distribution Γ_i (whose distribution function we denote again by Γ_i), such that $\Gamma_i(0) = 0$, and then jumps to type j with probability $m_{i,j}$. We write $\mu_i = \int_0^\infty x\, \Gamma_i(\mathrm{d}x)$ for the mean of the i^{th} lifetime, which can be infinite. Let $\overline{t}_j(t)$ be the time that the process spends at state j up to time t. Put $\overline{r}_{i,j}(t,a) = \mathbf{P}_i\left\{\bar{t}_j(t) \geq a\right\}$, $i, j \in \mathbf{K}$, where $\mathbf{P}_i$ stands for the probability when the process starts in type i. We aim at finding an upper bound for the probabilities $\overline{r}_{i,j}$.

First we show the connection between the Markov renewal process and the multitype branching system. We introduce the genealogical tree $\mathcal{T}$ of an individual, which comprises information on the individual's offspring genealogy, such as family relationships, mutations, death and birth times of individuals. For $t > 0$, let $\mathcal{T}_t$ denote the genealogical tree restricted to the time interval $[0, t]$. Finally, $\mathcal{T}_t^r$ stands for the reduced tree obtained from $\mathcal{T}_t$ by deleting the ancestry lines of those particles, which die before t. We write $\mathbf{P}_i$ for the law of $\mathcal{T}$, if the process started from an ancestor of type i with age 0. From the context it will be always clear when $\mathbf{P}_i$ refers to the branching particle system, or to the Markov renewal process.

For any given $t > 0$ and ancestry line $w \in \mathcal{T}_t$, let $t_j(w) \geq 0$ be the total time up to t that w spends in type $j \in \mathbf{K}$. Introduce the variable

$$\mu_j(t) = \min_{w \in \mathcal{T}_t^r} t_j(w), \tag{2.1}$$

which is the minimal time spent in type j among those particles that are alive at time t, with the usual convention that $\min \emptyset = \infty$. We also define the maximum spent time in type j up to time t:

$$\sigma_j(t) = \max_{w \in \mathcal{T}_t} t_j(w).$$

(Notice that, in this case, the population procreated by the ancestor is not necessarily alive at time t). Let

$$\nu_{i,j}(t,a) = \mathbf{P}_i\{\mu_j(t) \leq a\}$$

denote the probability that starting from i, there is a particle at time t, who spent less than a time in j. Note that for $t < a < \infty$,

$$\nu_{i,j}(t,a) = \mathbf{P}_i\{\mu_j(t) \leq a\} = \mathbf{P}_i\left\{\text{the process is not extinct at } t\right\} \to 0,$$

as $t \to \infty$, and for arbitrary $a < \infty$,

$$\nu_{i,j}(t,a) = \mathbf{P}_i\{\mu_j(t) \leq a\} \leq \mathbf{P}_i\left\{\text{the process is not extinct at } t\right\} \to 0,$$

as $t \to \infty$. Then by a renewal argument we obtain, for $a < t$, that

$$\begin{aligned} \nu_{i,i}(t,a) &= \int_0^a \Gamma_i(\mathrm{d}s)\left[1 - f_i(\mathbf{1} - \nu_{\cdot,i}(t-s, a-s))\right] \\ \nu_{i,j}(t,a) &= 1 - \Gamma_i(t) + \int_0^t \Gamma_i(\mathrm{d}s)\left[1 - f_i(\mathbf{1} - \nu_{\cdot,j}(t-s, a))\right]. \end{aligned} \tag{2.2}$$

Since $1 - f_i(\mathbf{1} - \mathbf{z}) \le \sum_{k=1}^K m_{i,k} z_k$, $\mathbf{z} = (z_1, \dots, z_K)$, we can compare the solution of (2.2) with the solution of the linear version

$$\begin{aligned} \alpha_{i,i}(t,a) &= \int_0^a \Gamma_i(\mathrm{d}s) \sum_{k=1}^K m_{i,k} \alpha_{k,i}(t-s, a-s) \\ \alpha_{i,j}(t,a) &= 1 - \Gamma_i(t) + \int_0^t \Gamma_i(\mathrm{d}s) \sum_{k=1}^K m_{i,k} \alpha_{k,j}(t-s, a). \end{aligned} \tag{2.3}$$

Notice that renewal argument implies again that $\alpha_{i,j}(t) = \mathbf{P}_i\{\bar{t}_j(t) \le a\}$ is the solution of the equation system (2.3). Let $\alpha_{i,j}^{(0)}(t,a) = \nu_{i,j}(t,a)$, and let $\alpha^{(n)} = (\alpha_{i,j}^{(n)})_{i,j=1,\dots,K}$, where

$$\begin{aligned} \alpha_{i,i}^{(n+1)}(t,a) &= \int_0^a \Gamma_i(\mathrm{d}s) \sum_{k=1}^K m_{i,k} \alpha_{k,i}^{(n)}(t-s, a-s) \\ \alpha_{i,j}^{(n+1)}(t,a) &= 1 - \Gamma_i(t) + \int_0^t \Gamma_i(\mathrm{d}s) \sum_{k=1}^K m_{i,k} \alpha_{k,j}^{(n)}(t-s, a). \end{aligned}$$

By induction it is clear that $\nu_{i,j}(t,a) \le \alpha_{i,j}^{(n)}(t,a)$ for all n. We show that the iteration converges to the solution $\alpha_{i,j}(t,a)$, and thus $\nu_{i,j}(t,a) \le \alpha_{i,j}(t,a)$. Let us fix a $t > 0$, and introduce the notation

$$||x - y||_t = \sup\{|x_{i,j}(s,u) - y_{i,j}(s,u)| \,:\, i,j \in \mathbf{K};\, 0 \le u < s \le t\}.$$

Then we get that, for all n,

$$||\alpha^{(n+1)} - \alpha^{(n)}||_t \le \max_{i \in \mathbf{K}} \int_0^t \sum_{k=1}^K m_{i,k} ||\alpha^{(n)} - \alpha^{(n-1)}||_t \Gamma_i(\mathrm{d}s) = \max_{i \in \mathbf{K}} \Gamma_i(t) ||\alpha^{(n)} - \alpha^{(n-1)}||_t,$$

where we used that the mean matrix M satisfies $\sum_{k=1}^K m_{i,k} = 1$ for all i. The last estimation implies convergence to the solution of (2.3); the proof of uniqueness of solutions of (2.3) follows in the same way. Therefore we showed that $\nu_{i,j}(t,a) \le \alpha_{i,j}(t,a)$. Notice that we only have shown that any solution of the equation system (2.2) is dominated by the *unique* solution of (2.3), which does not imply that (2.2) has a unique solution. We have proved:

Lemma 1 *For every $a \in (0,t)$ we have that*

$$\mathbf{P}_i\left\{\exists w \in \mathcal{T}_t^r \,:\, t_j(w) \le a\right\} \le \mathbf{P}_i\left\{\bar{t}_j(t) \le a\right\},$$

where the left side is for the branching process, while the right is for the Markov renewal process.

Exactly the same way as in [3] Lemma 10, we can show a similar bound.

Lemma 2 *For every $a \in (0,t)$ we have that*

$$\mathbf{P}_i\left\{\exists w \in \mathcal{T}_t : t_j(w) \ge a\right\} \le \mathbf{P}_i\left\{\bar{t}_j(t) \ge a\right\},$$

where the left side is for the branching process, while the right is for the Markov renewal process.

3 The Markov renewal process

In this section we are going to analyze the auxiliary Markov renewal process. First consider the discrete Markov chain $X_1, X_2, \ldots$ with transition matrix M, and let $\mathbf{p}^* = (p_1^*, \ldots, p_K^*)$ denote its stationary distribution. We have the following large deviation theorem for Markov chains ([5], Lemma 2.13.): For all $\delta > 0$ there exist positive constants C, c, such that

$$\mathbf{P}_i \left\{ \left| \frac{t_j(n)}{n} - p_j^* \right| > \delta \right\} \le C\mathrm{e}^{-cn}, \quad i \in \mathbf{K}, \tag{3.4}$$

where $\mathbf{P}_i$ stands for the probability measure, when the chain starts from position i, and $t_j(n)$ is the number of visits to state j among the first n steps:

$$t_j(n) = \#\{l : X_l = j, l = 1, 2, \ldots, n\}.$$

Here and below, several different constants arise in the calculations whose precise values are not relevant for our purposes. Therefore, for the reader's convenience we chose not to enumerate these constants. Hence the value of a constant may vary from line to line. In some proofs we use enumerated constants like $k_1, k_2, \ldots$, whose values are fixed only in the corresponding proof. Finally, we use some global constants $c_1, c_2, \ldots$, whose values are the same in the whole paper.

Let n_t denote the number of renewals up to time t. With these notations we may write

$$\bar{t}_j(t) = \xi_1^{(j)} + \xi_2^{(j)} + \cdots + \xi_{t_j(n_t)}^{(j)} + \eta_j(t) = S_{t_j(n_t)}^{(j)} + \eta_j(t), \tag{3.5}$$

where $\xi_1^{(j)}, \xi_2^{(j)}, \ldots$ are iid random variables with common distribution function Γ_j, and

$$\eta_j(t) = \begin{cases} t - Z_{n_t}, & \text{if } X_{n_t} = j, \\ 0, & \text{otherwise;} \end{cases}$$

that is $\eta_j(t)$ is non-zero only for one term, and stands for the spent lifetime. Here Z_n is the sum of the lifetimes up to the n^{th} renewal, and therefore Z_n is the sum of n independent, but not identically distributed random variables.

3.1 A long living particle type

Let $\gamma \in (0, 1]$. Assume that

$$\begin{aligned} &1 - \Gamma_1(x) \sim x^{-\gamma}, \quad \text{as } x \to \infty \text{ and} \\ &1 - \Gamma_j(x) \le A x^{-\eta_j}, \quad j = 2, 3, \ldots, K, \end{aligned} \tag{3.6}$$

where $A > 0$ and $\eta_j > 1$, $j = 2, 3, \ldots, K$. Put $\eta = \min\{\eta_j : j = 2, 3, \ldots, K\}$. We will show that with high probability the process spends $c\,t$ times in type 1.

Lemma 3 *There exists $c_1 > 0$ such that for every $i \in \mathbf{K}$ and $t > 1$*

$$\mathbf{P}_i \left\{ \frac{\bar{t}_1(t)}{t} \le c_1 \right\} \le C t^{1-\eta},$$

for some $C > 0$.

Proof. For simplicity we omit the lower index i. Recall that n_t stands for the number of renewals up to time t. For any $k_2 > 0$ we may write

$$\mathbf{P}\left\{\frac{\overline{t}_1(t)}{t} \le c_1\right\} = \mathbf{P}\left\{\frac{\overline{t}_1(t)}{t} \le c_1,\, n_t > k_2\, t\right\} + \mathbf{P}\left\{\frac{\overline{t}_1(t)}{t} \le c_1,\, n_t \le k_2\, t\right\}.$$

The first term is easy to estimate. Due to (3.4), with probability $\ge 1 - C\,\mathrm{e}^{-c\,t}$ we have $t_1(n_t)/n_t \ge p_1^*/2$, and so on this set

$$\frac{\overline{t}_1(t)}{t} \ge \frac{S^{(1)}_{t_1(n_t)}}{t_1(n_t)} \frac{t_1(n_t)}{n_t} \frac{n_t}{t} \ge \frac{S^{(1)}_{t_1(n_t)}}{t_1(n_t)} \frac{k_2\, p_1^*}{2}.$$

Truncation and Cramér's large deviation theorem shows that for any $d \in (0, \infty)$ there exist $C, c > 0$, such that for $n \in \mathbb{N}$,

$$\mathbf{P}\left\{\frac{S^{(1)}_n}{n} \le d\right\} \le C\,\mathrm{e}^{-c\,n}.$$

Applying this with $n \sim ct$, the estimation above shows that the first term $\le C\,\mathrm{e}^{-c\,t}$ (for some other pair of constants C, c) for any choice of c_1, k_2.

Now let us investigate the second term. Clearly $\overline{t}_1(t) \le c_1 t$ implies that $\overline{t}_j(t) > k_3 t$ for some $j \ge 2$, with $k_3 = (1 - c_1)/K$. If $n_t \le k_2 t$ then $\overline{t}_j(t) \le S^{(j)}_{t_j(n_t)+1} \le S^{(j)}_{\lfloor k_2 t \rfloor + 1}$ by (3.5), therefore the probability in question is less then

$$\mathbf{P}\left\{\frac{S^{(j)}_{\lfloor k_2 t \rfloor + 1}}{t} > k_3\right\} \le c\, t^{1-\eta_j} A,$$

which proves our lemma. In the last step we used Theorem 2 of Nagaev [10], which says that for any $c > 0$ and $x \ge cn$,

$$\mathbf{P}\left\{S^{(j)}_n - n\mu_j \ge x\right\} \le 2nx^{-\eta_j} A \tag{3.7}$$

for n large enough. In particular, for any $\delta > 0$,

$$\mathbf{P}\left\{\frac{S^{(j)}_n - n\mu_j}{n} \ge \delta\right\} \le cn^{1-\eta_j} A.$$

■

Notice that by using Lemma 6 below we obtain a stronger result. Namely, for any $\varepsilon > 0$

$$\mathbf{P}\left\{\frac{\overline{t}_1(t)}{t} \le c_1\right\} \le t^{\gamma+\varepsilon-\eta},$$

for t large enough. Combining this with Lemma 1 we obtain

Lemma 4 *For any $i \in \mathbf{K}$ there is a c_1 and $C > 0$ such that*

$$\mathbf{P}_i\left\{\frac{\mu_1(t)}{t} \le c_1\right\} \le Ct^{1-\eta}.$$

3.2 Occupation times for $j \geq 2$

To analyze the occupation times $\bar{t}_j(t)$ for $j = 2, \ldots, K$, we need a precise asymptotic for the number of renewals n_t.

We start by describing the asymptotic behavior of $S_n = \xi_1 + \cdots + \xi_n$, the sum of n independent random variables with distribution function Γ, for which $1 - \Gamma(x) \sim x^{-\gamma}$, $\gamma \in (0, 1]$. In the following, limits of sequences are meant as $n \to \infty$. We use the same convention for the continuous parameter t.

Lemma 5 *Assume that $d_n \to \infty$ if $\gamma < 1$, and $\log n/d_n \to 0$ when $\gamma = 1$. We have*

$$\mathbf{P}\{S_n > n^{1/\gamma} d_n\} \leq (1 + o(1)) d_n^{-\gamma}.$$

Moreover, for $\gamma < 1$ there exist constants c_γ such that for any sequences c_n for which $c_n \to 0$ and $n^{\gamma^{-1}-1} c_n \to \infty$, the following holds

$$\mathbf{P}\left\{S_n \leq c_n n^{1/\gamma}\right\} \leq 2 \exp\left\{-\frac{c_n^{-\frac{\gamma}{1-\gamma}}}{c_\gamma}\right\}.$$

If there exists a constant $L > 0$ such that $\sup_n n^{\gamma^{-1}-1} c_n < L$, $\gamma \in (0, 1]$, then for some $c > 0$

$$\mathbf{P}\left\{S_n \leq c_n n^{1/\gamma}\right\} \leq \mathrm{e}^{-c\,n},$$

for n large enough.

Note that in the case $\gamma = 1$ we can choose $c_n \equiv c > 0$ arbitrary large. We will use this remark in the proof of Lemma 6.

Proof. Let $\{t_n\}$ be a sequence of positive numbers such that $n[1 - \Gamma(t_n)] \to 0$ if $\gamma < 1$ (i.e. $t_n = n^{1/\gamma} d_n$ for some $d_n \to \infty$), and $n \log t_n / t_n \to 0$ when $\gamma = 1$ (that is, $t_n = n d_n$, where $\log n/d_n \to 0$). Using a theorem of Cline and Hsing ([1], Theorem 3.3) we get that

$$\lim_{n \to \infty} \sup_{s \geq t_n} \left| \frac{\mathbf{P}\{S_n > s\}}{n[1 - \Gamma(s)]} - 1 \right| = 0.$$

This follows immediately from [1] if $\gamma < 1$, while for $\gamma = 1$ one has to check that the sequence which has to converge to 0, is

$$\frac{n}{t_n} \int_1^{t_n} x \mathrm{d}\Gamma(x) \sim \frac{n \log t_n}{t_n}.$$

Writing $s = t_n$ in the form $t_n = n^{1/\gamma} d_n$ we obtain the first statement.

Now we turn to the upper estimates for $S_n / n^{1/\gamma}$.

We use a truncation method. For $a > 0$ let denote

$$\xi^{(a)} = \begin{cases} \xi, & \text{if } \xi \leq a, \\ a, & \text{otherwise,} \end{cases}$$

the truncated variable at a. For the first moment of this variable, as $a \to \infty$ we have

$$\mu_a = \mathbf{E}\xi^{(a)} = \int_0^a x \mathrm{d}\Gamma(x) + a[1-\Gamma(a)] \sim \begin{cases} \frac{1}{1-\gamma} a^{1-\gamma}, & \text{for } \gamma \neq 1, \\ \log a, & \text{for } \gamma = 1. \end{cases}$$

For the proof we need Bernstein's inequality (cf. p. 855 of Shorack and Wellner [11]). Let $X_1, X_2, \dots$ be iid random variables with $\mathbf{E}X_1 = 0$, and let $\kappa > 0$ and $v > 0$ be constants such that $\mathbf{E}|X^m| \leq v\kappa^{m-2}m!/2$. Then for the partial sum $S_n = X_1 + \cdots + X_n$ the following holds:

$$\mathbf{P}\left\{|S_n| > t\right\} \leq 2 \exp\left\{-\frac{t^2}{2vn + 2\kappa t}\right\}. \tag{3.8}$$

Easy computations show that in our case (that is if $1 - \Gamma(x) \sim x^{-\gamma}$) for $m \geq 2$

$$\mathbf{E}[\xi^{(a)}]^m \sim \frac{m}{m-\gamma} a^{m-\gamma}, \quad \text{as } a \to \infty.$$

Since $\mathbf{E}|\xi^{(a)} - \mu_a|^m \leq \mathbf{E}[\xi^{(a)}]^m + \mu_a^m$, this shows that in Bernstein's inequality (3.8) we can choose $v = 2\,a^{2-\gamma}$ and $\kappa = a$. Obviously $\mathbf{P}\left\{S_n \leq c_n n^{1/\gamma}\right\} \leq \mathbf{P}\left\{S_n^{(a)} \leq c_n n^{1/\gamma}\right\}$, and so we may write

$$\mathbf{P}\left\{S_n \leq c_n n^{1/\gamma}\right\} \leq \mathbf{P}\left\{S_n^{(a)} - n\mu_a \leq c_n n^{1/\gamma} - n\mu_a\right\} \leq \mathbf{P}\left\{|S_n^{(a)} - n\mu_a| \geq n\mu_a - c_n n^{1/\gamma}\right\},$$

where in the last inequality we assumed that $n\mu_a > c_n n^{1/\gamma}$. Applying Bernstein's inequality with $v = 2\,a^{2-\gamma}$ and $\kappa = a$ we obtain

$$\mathbf{P}\left\{S_n \leq c_n n^{1/\gamma}\right\} \leq 2 \exp\left\{-\frac{(n\mu_a - c_n n^{1/\gamma})^2}{4a^{2-\gamma}n + 2a(n\mu_a - c_n n^{1/\gamma})}\right\}. \tag{3.9}$$

Let $n\mu_a = 2n^{1/\gamma}c_n$, that is $a \sim [2(1-\gamma)]^{\frac{1}{1-\gamma}}\, n^{\frac{1}{\gamma}} c_n^{\frac{1}{1-\gamma}} =: a_n$. By our assumptions, a_n tends to ∞. Then the numerator in the exponential of (3.9) is $n^{2/\gamma}c_n^2$, while the denominator is

$$4([2(1-\gamma)]^{\frac{1}{1-\gamma}}\, n^{\frac{1}{\gamma}} c_n^{\frac{1}{1-\gamma}})^{2-\gamma} n + 2([2(1-\gamma)]^{\frac{1}{1-\gamma}}\, n^{\frac{1}{\gamma}} c_n^{\frac{1}{1-\gamma}}) c_n n^{1/\gamma} = c_\gamma\, n^{2/\gamma} c_n^{\frac{2-\gamma}{1-\gamma}},$$

with $c_\gamma = (10 - 8\gamma)[2 - 2\gamma]^{\frac{1}{1-\gamma}}$. In this way we obtain finally that

$$\mathbf{P}\left\{S_n \leq c_n n^{1/\gamma}\right\} \leq 2 \exp\left\{-\frac{c_n^{-\frac{\gamma}{1-\gamma}}}{c_\gamma}\right\},$$

which is the desired bound. The last assertion in the lemma follows easily from Cramér's large deviation theorem, together with the truncation method. ■

Next we investigate the asymptotic behavior of the number of renewals n_t in our Markov renewal process, where the lifetime distributions $\Gamma_1, \dots, \Gamma_K$ are as in (3.6). We show that n_t asymptotically behaves like the number of renewals $\tilde{n}_t$ in a *standard* renewal process, where the tail of the lifetime distribution is $\sim x^{-\gamma}$.

Lemma 6 *Let $c(\cdot)$ be a function such that*

$$\lim_{t\to\infty} c(t) = 0 \text{ if } \gamma < 1, \quad \text{and} \quad \lim_{t\to\infty} c(t)\log t = 0 \text{ if } \gamma = 1. \tag{3.10}$$

Then for all $i \in \mathbf{K}$, for $t > 0$ large enough,

$$\mathbf{P}_i\left\{\frac{n_t}{t^\gamma} \le c(t)\right\} \le 2c(t), \quad 0 < \gamma \le 1.$$

If $\gamma < 1$ then for any $a \in (0, 1-\gamma)$, for all $i \in \mathbf{K}$ and all $t > 0$ large enough,

$$\mathbf{P}_i\left\{\frac{n_t}{t^\gamma} > t^a\right\} \le 2\,\exp\left\{-c\,t^{\frac{a}{1-\gamma}}\right\}.$$

If $\gamma = 1$, then for any $a > 0$ and $i \in \mathbf{K}$ there is a constant $c > 0$ such that for all $t > 0$ large enough,

$$\mathbf{P}_i\left\{\frac{n_t}{t} > a\right\} \le \mathrm{e}^{-ct}.$$

Proof. We drop the lower index i. In order to get an upper bound for n_t let us define

$$\tilde{\Gamma}(x) = \prod_{i=1}^{K} \Gamma_i(x)\,,$$

which is the distribution function of the lifetime $\tilde{\xi} \stackrel{\mathcal{D}}{=} \max\{\xi^{(1)}, \ldots, \xi^{(K)}\}$, where $\xi^{(1)}, \ldots, \xi^{(K)}$ are independent and distributed as $\Gamma_1, \ldots, \Gamma_K$ respectively. Consider a standard renewal process $\tilde{n}_t$, $\tilde{S}_n$ with this lifetime distribution. Recall that Z_n is the sum of the lifetimes up to the n^{th} renewal. Clearly

$$\mathbf{P}\left\{n_t \le t^\gamma c(t)\right\} = \mathbf{P}\left\{Z_{\lfloor t^\gamma c(t)\rfloor} > t\right\} \le \mathbf{P}\left\{\tilde{S}_{\lfloor t^\gamma c(t)\rfloor} > t\right\}.$$

According to Lemma 15 below, $1 - \tilde{\Gamma}(x) \sim x^{-\gamma}$. Therefore using Lemma 5 we can write

$$\mathbf{P}\left\{\tilde{S}_{\lfloor t^\gamma c(t)\rfloor} > t\right\} \le 2\,c(t),$$

where we have assumed that for $\gamma < 1$ the convergence $t/[t^\gamma c(t)]^{1/\gamma} \to \infty$ holds, which is equivalent to $c(t) \to 0$, and that $c(t)\log t \to 0$ when $\gamma = 1$. This proves the first statement.

To obtain the lower bound we use the simple estimation $Z_n \ge S^{(1)}_{t_1(n)}$, that is, we simply drop the lifetimes with finite mean. First consider the case $\gamma < 1$. Then we have

$$\begin{aligned}
\left\{Z_{\lfloor t^{\gamma+a}\rfloor} \le t\right\} &\subset \left\{S^{(1)}_{t_1(\lfloor t^{\gamma+a}\rfloor)} \le t\right\} \\
&\subset \left\{S^{(1)}_{\lfloor (p_1^*-\varepsilon)t^{\gamma+a}\rfloor} \le t,\ \frac{t_1(\lfloor t^{\gamma+a}\rfloor)}{\lfloor t^{\gamma+a}\rfloor} > p_1^* - \frac{\varepsilon}{2}\right\} \cup \left\{\frac{t_1(\lfloor t^{\gamma+a}\rfloor)}{\lfloor t^{\gamma+a}\rfloor} \le p_1^* - \frac{\varepsilon}{2}\right\},
\end{aligned}$$

hence using Lemma 5 and (3.4) we have

$$\begin{aligned}
\mathbf{P}\left\{\frac{n_t}{t^\gamma} > t^a\right\} &= \mathbf{P}\left\{n_t > \lfloor t^{\gamma+a}\rfloor\right\} = \mathbf{P}\left\{Z_{\lfloor t^{\gamma+a}\rfloor} \le t\right\} \\
&\le \mathbf{P}\left\{S^{(1)}_{\lfloor (p_1^*-\varepsilon)t^{\gamma+a}\rfloor} \le t\right\} + c\,\mathrm{e}^{-ct^{\gamma+a}} \\
&= \mathbf{P}\left\{\frac{S^{(1)}_{\lfloor (p_1^*-\varepsilon)t^{\gamma+a}\rfloor}}{\lfloor (p_1^*-\varepsilon)t^{\gamma+a}\rfloor^{1/\gamma}} \le \frac{t}{\lfloor (p_1^*-\varepsilon)t^{\gamma+a}\rfloor^{1/\gamma}}\right\} + c\,\mathrm{e}^{-ct^{\gamma+a}} \\
&\le \mathbf{P}\left\{\frac{S^{(1)}_{\lfloor (p_1^*-\varepsilon)t^{\gamma+a}\rfloor}}{\lfloor (p_1^*-\varepsilon)t^{\gamma+a}\rfloor^{1/\gamma}} \le c\,t^{-a/\gamma}\right\} + c\,\mathrm{e}^{-ct^{\gamma+a}} \\
&\le 2\,\exp\left\{-c\,t^{\frac{a}{1-\gamma}}\right\} + c\,\mathrm{e}^{-ct^{\gamma+a}}.
\end{aligned}$$

Taking into account that $\gamma + a > a/(1-\gamma)$, we obtain the statement. Finally, when $\gamma = 1$ we use exactly the same method. Using the event-decomposition as before, the last part of Lemma 5 and (3.4) we have

$$\mathbf{P}\left\{\frac{n_t}{t} > a\right\} \le \mathbf{P}\left\{S^1_{\lfloor a(p_1^*-\varepsilon)t\rfloor} \le t\right\} + c\,\mathrm{e}^{-ct} \le c\,\mathrm{e}^{-c\,t},$$

thus proving the last assertion of the lemma. ∎

The preceding results allow us to obtain the following estimations for the probabilities of the smallness and largeness of $\bar{t}_j(t)/t^\gamma$.

Lemma 7 *Assume that* (3.6) *holds, and let* $c(\cdot)$ *be a function satisfying* (3.10)*. Then for all* $i \in \mathbf{K}$*, any* $j = 2, 3, \ldots, K$*, every* $\varepsilon > 0$ *and all* t *large enough,*

$$\mathbf{P}_i\left\{\bar{t}_j(t) \le t^\gamma c(t)\right\} \le c\,(c(t) + t^{\varepsilon-\gamma}), \quad 0 < \gamma \le 1.$$

If $\gamma < 1$ *then for any* $0 < a < 1-\gamma$*,*

$$\mathbf{P}_i\left\{\bar{t}_j(t) \ge t^{\gamma+a}\right\} \le t^{1-\eta\gamma},$$

while for $\gamma = 1$*,*

$$\mathbf{P}_i\left\{\bar{t}_j(t) \ge a\,t\right\} \le c\,t^{1-\eta}$$

for any $a > 0$*.*

Proof. As before, for simplicity we omit the lower index i. Clearly, for any $\varepsilon > 0$,

$$\mathbf{P}\left\{\bar{t}_j(t) \le t^\gamma c(t)\right\} \le \mathbf{P}\left\{\bar{t}_j(t) \le t^\gamma c(t), n_t \ge t^\varepsilon\right\} + \mathbf{P}\left\{n_t < t^\varepsilon\right\}.$$

Due to (3.4), on the set $\{n_t \ge t^\varepsilon\}$ we have, for any $\delta > 0$, that $t_j(n_t)/n_t \in (p_j^* - \delta, p_j^* + \delta)$ with probability $\ge 1 - c\,\mathrm{e}^{-c\,t^\varepsilon}$. Truncation method and Cramér's large deviation theorem show that

$$\mathbf{P}\left\{\frac{S^{(j)}_{t_j(n_t)}}{t_j(n_t)} < \frac{\mu_j}{2}, n_t \ge t^\varepsilon\right\} \le c\,\mathrm{e}^{-c\,t^\varepsilon}.$$

Since

$$\frac{\overline{t}_j(t)}{t^\gamma} = \frac{S^{(j)}_{t_j(n_t)} + \eta_j(t)}{t_j(n_t)} \frac{t_j(n_t)}{n_t} \frac{n_t}{t^\gamma} \tag{3.11}$$

we have obtained that

$$\mathbf{P}\left\{\overline{t}_j(t) \le t^\gamma c(t), n_t \ge t^\varepsilon\right\} \le \mathbf{P}\left\{n_t \le c\, t^\gamma c(t)\right\} + c\,\mathrm{e}^{-c\,t^\varepsilon} \le c\,(c(t) + \mathrm{e}^{-c\,t^\varepsilon}),$$

where in the last step we used Lemma 6. Taking into account that $\mathbf{P}\left\{n_t < t^\varepsilon\right\} \le c\,t^{\varepsilon-\gamma}$, which follows again from Lemma 6, the first inequality is proved.

For the second part we use a similar technique. We first deal with the case $\gamma < 1$. For any $0 < b < 1$ we may write

$$\mathbf{P}\left\{\overline{t}_j(t) \ge t^{\gamma+a}\right\} = \mathbf{P}\left\{\overline{t}_j(t) \ge t^{\gamma+a}, n_t \ge t^b\right\} + \mathbf{P}\left\{\overline{t}_j(t) \ge t^{\gamma+a}, n_t < t^b\right\}.$$

If $n_t < t^b$, using that $S^{(j)}_{t_j(n_t)} + \eta_j(t) \le S^{(j)}_{t_j(n_t)+1}$ we get $\overline{t}_j(t) \le S^{(j)}_{t_j(n_t)+1} \le S^{(j)}_{\lfloor t^b \rfloor+1}$. Therefore the tail probabilities of the right-hand term satisfy the inequality

$$\mathbf{P}\left\{S^{(j)}_{\lfloor t^b \rfloor+1} \ge t^{\gamma+a}\right\} \le 2^{1+\eta_j}\left(\lfloor t^b \rfloor + 1\right) t^{-\eta_j(\gamma+a)} A \le t^{1-\eta\gamma} \tag{3.12}$$

for all t large enough, where we used again Nagaev's result (3.7). Note that we only needed that $\gamma + a > b$.

For the estimation of the other term we use again the decomposition (3.11). Since $n_t \ge t^b$, by (3.4) we have $t_j(n_t)/n_t \in (p_j^*/2, 2p_j^*)$ with probability $\ge 1 - c\,\mathrm{e}^{-c\,t^b}$. Due to Lemma 6, for any $\varepsilon' < a/2$

$$\mathbf{P}\left\{\frac{n_t}{t^\gamma} \ge t^{\varepsilon'}\right\} \le 2\,\exp\left\{-ct^{\frac{\varepsilon'}{1-\gamma}}\right\}.$$

Since the orders of these terms are smaller than that of $t^{1-\eta\gamma}$ (given in the statement), we can work on $\{t^b \le n_t < t^{\gamma+\varepsilon'}\} \cap \{t_j(n_t)/n_t \in (p_j^*/2, 2p_j^*)\}$. On this event, by (3.11)

$$\frac{\overline{t}_j(t)}{t^\gamma} \le \frac{S^{(j)}_{t_j(n_t)+1}}{t_j(n_t)} \frac{t_j(n_t)}{n_t} \frac{n_t}{t^\gamma} \le \frac{S^{(j)}_{t_j(n_t)+1}}{t_j(n_t)} 2p_j^* t^{\varepsilon'},$$

and so $\overline{t}_j(t) \ge t^{\gamma+a}$ implies $S^{(j)}_{t_j(n_t)+1}/t_j(n_t) \ge t^{a-\varepsilon'}/(2p_j^*)$. Thus for t large enough

$$\begin{aligned}
&\mathbf{P}\left\{\frac{t_j(t)}{t^\gamma} > t^a, \frac{t_j(n_t)}{n_t} \in (p_j^*/2, 2p_j^*), t^b \le n_t \le t^{\gamma+\varepsilon'}\right\} \\
&\le \mathbf{P}\left\{\frac{S^{(j)}_{t_j(n_t)+1}}{t_j(n_t)} \ge t^{a-\varepsilon'}, t_j(n_t) \ge t^b p_j^*/2\right\} \\
&\le 2\,t^{1-\eta(a+b-\varepsilon')},
\end{aligned}$$

where the last inequality follows again from (3.7). Choosing b such that $a + b > \gamma$ and $\varepsilon' < a + b - \gamma$ we obtain the desired order $t^{1-\eta\gamma}$. This, together with (3.12) gives the statement.

The proof in the case $\gamma = 1$ follows a similar approach. For any $b > 0$

$$\mathbf{P}\left\{\bar{t}_j(t) \geq a\,t\right\} \leq \mathbf{P}\left\{\bar{t}_j(t) \geq a\,t, n_t \leq b\,t\right\} + \mathbf{P}\left\{n_t > b\,t\right\}.$$

The second summand in the right of the above inequality is exponentially small for any $b > 0$, and we have already shown in the proof of Lemma 3 that the first one is less than $ct^{1-\eta}$, provided that b is small enough. ■

Combining the last result with Lemma 1 and Lemma 2 respectively we obtain

Lemma 8 *Assume that* (3.6) *holds and that* $c(\cdot)$ *is a function satisfying* (3.10). *Then for all* $i \in \mathbf{K}$, *all* $j = 2, 3, \ldots, K$, *each* $\varepsilon > 0$ *and all* t *large enough,*

$$\mathbf{P}_i\left\{\exists w \in \mathcal{T}_t^r : t_j(w) \leq t^\gamma c(t)\right\} = \mathbf{P}_i\left\{\mu_j(t) \leq t^\gamma c(t)\right\} \leq c\,(c(t) + t^{\varepsilon-\gamma}).$$

If $\gamma < 1$, *then for any* $0 < a < 1 - \gamma$

$$\mathbf{P}_i\left\{\exists w \in \mathcal{T}_t : t_j(w) \geq t^{\gamma+a}\right\} = \mathbf{P}_i\left\{\sigma_j(t) \geq t^{\gamma+a}\right\} \leq t^{1-\eta\gamma},$$

while if $\gamma = 1$, *then for any* $a > 0$

$$\mathbf{P}_i\left\{\exists w \in \mathcal{T}_t : t_j(w) \geq a\,t\right\} = \mathbf{P}_i\left\{\sigma_j(t) \geq a\,t\right\} \leq c\,t^{1-\eta}.$$

4 Extinction results

In this final section, we apply the results on occupation times proved earlier in the paper to analyze extinction properties of our branching particle system. Let N_t denote the particle system at time t, i.e. N_t is the point measure on $\mathbb{R}^d \times \mathbf{K}$ determined by the positions and types of individuals alive at time $t \geq 0$. We write N_t^i for the point measure representing the population of type-i particles at time t, that is $N_t^i(A) = N_t(A \times \{i\})$ for any $A \subset \mathbb{R}^d$, hence $N_t = N_t^1 + \cdots + N_t^K$. As before the lower indices in $\mathbf{P}$ and $\mathbf{E}$ refer to the initial distribution. In particular, $\mathbf{P}_{x,i}$ and $\mathbf{E}_{x,i}$ refer to a population having an ancestor $\delta_{(x,i)}$ of type $i \in \mathbf{K}$, initially at position $x \in \mathbb{R}^d$.

Let $\mathbf{h} : \mathbb{R}^d \times \mathbf{K} \to [0, \infty)$ be continuous function with compact support. We write $\langle \mu, \mathbf{h}\rangle = \int \mathbf{h}\,\mathrm{d}\mu$ for any measure μ on $\mathcal{B}(\mathbb{R}^d \times \mathbf{K})$. Without danger of confusion we also write $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^K x_i y_i$ for the scalar product of vectors $\mathbf{x} = (x_1, \ldots, x_K)$ and $\mathbf{y} = (y_1, \ldots, y_K)$. Assume that the initial population N_0 is a Poisson process with intensity measure $\Lambda = \lambda_1\,\ell\delta_{\{1\}} + \cdots + \lambda_K\,\ell\delta_{\{K\}}$, where ℓ is d-dimensional Lebesgue measure, and λ_i, $i \in \mathbf{K}$, are non-negative constants.

The Laplace transform of our branching process is, for any $t \geq 0$, given by

$$\begin{aligned} \mathbf{E}\left[\mathrm{e}^{-\langle N_t, \mathbf{h}\rangle}\right] &= \exp\left\{-\sum_{j=1}^K \lambda_j \int_{\mathbb{R}^d} \mathbf{E}_{x,j}\left[1 - \mathrm{e}^{-\langle N_t, \mathbf{h}\rangle}\right] \mathrm{d}x\right\} \\ &= \exp\left\{-\left\langle \Lambda, \mathbf{1} - \mathbf{E}_{\cdot,\cdot}\mathrm{e}^{-\langle N_t, \mathbf{h}\rangle}\right\rangle\right\}. \end{aligned}$$

We put $U_i(\mathbf{h}, t, x) = \mathbf{E}_{x,i}\left(1 - \mathrm{e}^{-\langle N_t, \mathbf{h}\rangle}\right)$.

To prove extinction of $\{N_t,\, t \geq 0\}$ it suffices to show that the Laplace transform of N_t converges to the Laplace transform of the empty population, and for this it is enough to verify that

$$\langle \Lambda, U_{\cdot}(\mathbf{h}, t, \cdot)\rangle \to 0 \quad \text{as } t \to \infty,$$

which is the same as

$$U_i^+(\mathbf{h}, t) := \int \left[\mathbf{E}_{x,i}\left(1 - \mathrm{e}^{-\langle N_t, \mathbf{h}\rangle}\right)\right] \mathrm{d}x \to 0 \quad \text{as } t \to \infty \text{ for all } i \in \mathbf{K}.$$

Let $B \subset \mathbb{R}^d$ be a ball, and assume that $B \times \mathbf{K} \supset \operatorname{supp}\mathbf{h}$. Then

$$1 - \mathrm{e}^{-\langle N_t, \mathbf{h}\rangle} \leq I(N_t(B \times \mathbf{K}) > 0),$$

which implies

$$\mathbf{E}_{x,i}\left[1 - \mathrm{e}^{-\langle N_t, \mathbf{h}\rangle}\right] \leq \mathbf{P}_{i,x}\left\{N_t(B \times \mathbf{K}) > 0\right\}.$$

Conversely, if $\mathbf{h}|_{B \times \mathbf{K}} \geq 1$, then

$$1 - \mathrm{e}^{-\langle N_t, \mathbf{h}\rangle} \geq (1 - \mathrm{e}^{-1}) I(N_t(B \times \mathbf{K}) > 0),$$

and so

$$\mathbf{E}_{x,i}\left[1 - \mathrm{e}^{-\langle N_t, \mathbf{h}\rangle}\right] \geq (1 - \mathrm{e}^{-1})\mathbf{P}_{i,x}\left\{N_t(B \times \mathbf{K}) > 0\right\}.$$

In this way we get that

Lemma 9 *Extinction of $\{N_t,\, t \geq 0\}$ occurs if, and only if for any bounded Borel set $B \subset \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \mathbf{P}_{i,x}\left\{N_t(B \times \mathbf{K}) > 0\right\} \mathrm{d}x \to 0 \textit{ for all } i \in \mathbf{K}, \textit{ as } t \to \infty.$$

Put $\alpha = \min\{\alpha_i : i = 1, 2, \ldots, K\}$. Recall the following result from [3]:

Lemma 10 *(Fleischmann & Vatutin). For each bounded $B \subset \mathbb{R}^d$*

$$\sup_{t \geq 1} \int_{\mathbb{R}^d \setminus C(t,L)} \mathbf{E}_{x,i} N_t(B \times \mathbf{K}) \mathrm{d}x \longrightarrow 0 \textit{ as } L \uparrow \infty,$$

where $C(t, L) = \{x \in \mathbb{R}^d : |x| \leq L t^{1/\alpha}\}$.

This means that extinction of $\{N_t,\, t \geq 0\}$ occurs if, and only if for any bounded Borel set $B \subset \mathbb{R}^d$, and for L large enough

$$\int_{C(t,L)} \mathbf{P}_{i,x}\left\{N_t(B \times \mathbf{K}) > 0\right\} \mathrm{d}x \to 0 \text{ for all } i \in \mathbf{K} \text{ as } t \to \infty. \tag{4.13}$$

4.1 Lifetimes with finite means

When the lifetimes have finite mean and the dimension is small, it is not necessary to analyze the occupation times in order to prove local extinction. As we are going to show, in this case a simple estimation and the asymptotics of the extinction probabilities of critical multitype branching processes give the result.

Let $F^{(i)}$ denote the probability generating function of the process starting from a single particle of type i:

$$F^{(i)}(t; s_1, \ldots, s_K) = \mathbf{E}_i \left[s_1^{N_t^1(\mathbb{R}^d)} \cdots s_K^{N_t^K(\mathbb{R}^d)} \right], \quad 0 \le s_j \le 1,\ j \in \mathbf{K}. \tag{4.14}$$

Put $Q^{(i)}(t; s_1, \ldots, s_K) = 1 - F^{(i)}(t; s_1, \ldots, s_K)$ and $q^{(i)}(t; s) = Q^{(i)}(t; s, \ldots, s)$. Clearly

$$\mathbf{P}_{i,x} \left\{ N_t(B \times \mathbf{K}) > 0 \right\} \le \mathbf{P}_i \left\{ \text{the process is not extinct at time } t \right\}.$$

Consider the discrete–time multitype Galton–Watson process $\{\mathbf{X}_n\}$, with the same offspring distributions as in the branching particle system. Let $\mathbf{v}$ and $\mathbf{u}$ respectively denote the left and right normed eigenvectors of the mean matrix M, which are determined by:

$$\mathbf{v}M = \mathbf{v}, \quad M\mathbf{u} = \mathbf{u}, \quad \mathbf{vu} = 1, \quad \mathbf{1u} = 1. \tag{4.15}$$

Since by assumption M is stochastic, $\mathbf{u} = K^{-1}\mathbf{1}$. Let $\mathbf{f}_n = (f_n^1, \ldots, f_n^K)$ denote the generating function of the n^{th} generation, that is $f_n^i(\mathbf{x}) = \mathbf{E}_i \left[\mathbf{x}^{\mathbf{X}_n} \right]$ and put $\mathbf{f}_1(\mathbf{x}) = \mathbf{f}(\mathbf{x})$. It is well-known that $\mathbf{f}_{n+1}(x) = \mathbf{f}(\mathbf{f}_n(\mathbf{x}))$. Let us assume that

$$x - \langle \mathbf{v}, \mathbf{1} - \mathbf{f}(\mathbf{1} - \mathbf{u}\,x) \rangle \sim x^{1+\beta} L(x) \quad \text{as } x \to 0, \tag{4.16}$$

where $\beta \in (0,1]$ and L is slowly varying at 0 in the sense that $\lim_{x \to 0} L(\lambda x)/L(x) = 1$ for every $\lambda > 0$. In this case, for the survival probabilities it is known that

$$\mathbf{1} - \mathbf{f}_n(0) = (\mathbf{u} + o(1)) n^{-1/\beta} L_1(n) \text{ as } n \to \infty,$$

where L_1 is slowly varying at ∞ (see Theorem 1 in [12] or Theorem 1 in [13]). Moreover, assume that

$$\lim_{n \to \infty} \frac{n[1 - \Gamma_i(n)]}{\langle \mathbf{v}, \mathbf{1} - \mathbf{f}_n(0) \rangle} = 0, \quad i = 1, 2, \ldots, K. \tag{4.17}$$

Then

$$Q^{(i)}(t; 0) = \mathbf{P}_i \left\{ \text{the process is not extinct at } t \right\} \sim u_i\, D^{\frac{1}{\beta}} t^{-\frac{1}{\beta}} L_1(t) \text{ as } t \to \infty, \tag{4.18}$$

where $D = \sum_{i=1}^K u_i v_i \mu_i$; see Theorem 2 in [13].

Using the estimate above, we obtain the following theorem.

Theorem 1 *Assume that* (4.16) *and* (4.17) *hold. Then for $d < \alpha/\beta$ the process* $\{N_t,\ t \ge 0\}$ *suffers local extinction.*

Proof. Due to (4.18), for any $\varepsilon > 0$

$$Q^{(i)}(t;0) \leq c\,t^{-\frac{1-\varepsilon}{\beta}}, \quad i \in \mathbf{K}.$$

Plugging this into (4.13) we get

$$\int_{C(t,L)} \mathbf{P}_{i,x}\left\{N_t(B \times \mathbf{K}) > 0\right\} \mathrm{d}x \leq c\,t^{\frac{d}{\alpha}-\frac{1-\varepsilon}{\beta}}.$$

Since by assumption $d < \alpha/\beta$, for some $\varepsilon > 0$ the exponent of t in the above inequality is negative, which implies that the integral in the left-hand side tends to 0. ■

Remark 1. When the generating functions f_i, $i \in \mathbf{K}$, are of the form $f_i(s,\ldots,s) = f_i(s) = s + c\,(1-s)^{1+\beta_i}$ where $\beta_i \in (0,1]$, it is easy to verify that (4.16) holds with $\beta = \min\{\beta_i : i \in \mathbf{K}\}$, and that (4.17) is fulfilled if for some $\varepsilon > 0$

$$\lim_{n\to\infty} n^{1+\frac{1}{\beta}+\varepsilon}[1 - \Gamma_i(n)] = 0, \quad i = 1, 2, \ldots, K.$$

Remark 2. We remark that we do not need the precise asymptotic decay of the non-extinction probabilities given in (4.18); it suffices to know an asymptotic order of decay. In order to get this, instead of assuming in (4.16) that $L(\cdot)$ is slowly varying at 0, it is enough to suppose that L is an *S–O varying function*, meaning that there exists an $A > 0$ such that $\limsup_{x\to 0} L(\lambda x)/L(x) < A$ for any $\lambda > 0$. S–O varying functions were introduced by Drasin and Seneta [2]. The definition immediately implies that $\liminf_{x\to 0} L(\lambda x)/L(x) > A^{-1}$ for all $\lambda > 0$. It was shown in [2] that every S–O varying function admits a representation as the product of a slowly varying function and a bounded (away from 0 and ∞) function. A careful analysis of the proof of Theorem 1 in [12] shows that, under the S–O varying assumption on L, we have that for any $\varepsilon > 0$ and for all n large enough,

$$|\mathbf{1} - \mathbf{f}_n(0)| \leq n^{-\frac{1-\varepsilon}{\beta}}.$$

Since this estimate is precisely what we use in the proofs of our extinction theorems, all these results (including the infinite mean case) remain true in this more general setup. If (4.17) holds (which in particular implies that the lifetimes have finite mean), we obtain that for any $\varepsilon > 0$ and for all t large enough,

$$Q^{(i)}(t;0) = \mathbf{P}_i\left\{\text{the process is not extinct at } t\right\} \leq t^{-\frac{1-\varepsilon}{\beta}}.$$

4.2 A lifetime with infinite mean – Case A

From now on we assume that there is exactly one lifetime distribution with infinite mean; more precisely we assume (3.6). Moreover, in this subsection we additionally assume that $\alpha = \min\{\alpha_i : i \in \mathbf{K}\} = \alpha_1$, that is, the long-living particle type is the most mobile as well.

In the following, $\mathbf{P}^{\theta}_{x,i}$ denotes the distribution of the population starting with a single individual $\delta_{(x,i)}$ of age $\theta \geq 0$.

Lemma 11 *For all $(x,i) \in \mathbb{R}^d \times \mathbf{K}$, all bounded Borel $B \subset \mathbb{R}^d$ and all $t > 0$,*

$$\mathbf{P}^{\theta}_{x,i}\left\{N_t(B \times \mathbf{K}) > 0\right\} \le c_2 \left(t^{-d/\alpha} + t^{1-\eta}\right),$$

where the constant c_2 is independent of θ, x and i.

Proof. Put $A = \{\mu_1(t) \le c_1 t\}$, i.e. A is the event that there exists a branch $w \in \mathcal{T}^r_t$ such that $t_1(w) \le c_1\, t$, so the process spends less than $c_1\, t$ time in type 1 for some branch w. Clearly, due to Lemma 4, we may write

$$\begin{aligned} \mathbf{P}\{N_t(B \times \mathbf{K}) > 0\} &\le \mathbf{P}\{A\} + \mathbf{P}\{N_t(B \times \mathbf{K}) > 0, A^c\} \\ &\le ct^{1-\eta} + \mathbf{E}\left[N_t(B \times \mathbf{K}) I_{A^c}\right]. \end{aligned}$$

Conditioning on the reduced tree and noting that A is $\mathcal{T}^r_t$ measurable we have

$$\begin{aligned} \mathbf{E}^{\theta}_{x,i}\left[N_t(B \times \mathbf{K}) I_{A^c}\right] &= \mathbf{E}^{\theta}_{x,i} \sum_{j=1}^{K} \sum_{l=1}^{N^j_t(\mathbb{R}^d)} I_{A^c} I(W^l_j(t) \in B) \\ &= \mathbf{E}^{\theta}_{x,i}\mathbf{E}^{\theta}_{x,i}\left[\sum_{j=1}^{K} \sum_{l=1}^{N^j_t(\mathbb{R}^d)} I_{A^c} I(W^l_j(t) \in B) \Big| \mathcal{T}^r_t\right] \\ &= \mathbf{E}^{\theta}_{x,i}\left[I_{A^c} \sum_{j=1}^{K} \sum_{l=1}^{N^j_t(\mathbb{R}^d)} \mathbf{P}^{\theta}_{x,i}\left\{W^l_j(t) \in B \big| \mathcal{T}^r_t\right\}\right], \end{aligned}$$

where, given $\mathcal{T}^r_t$,

$$W^l_j(t) \stackrel{\mathcal{D}}{=} W(t_1, \alpha_1) + \cdots + W(t_K, \alpha_K). \tag{4.19}$$

Here t_j is the time that a branch of the reduced tree spent in type j, $t_1 + \cdots + t_K = t$, and $\{W(t, \alpha_j),\ t \ge 0\}$ are independent symmetric α_j-stable motions starting from 0, $j = 1, \ldots, K$. Since on the complement of A any branch spent at least $c_1 t$ time in type 1, we have

$$\begin{aligned} \mathbf{P}^{\theta}_{x,i}\left\{W^l_j(t) \in B;\ A^c \big| \mathcal{T}^r_t\right\} &= \int p_{t-t_1}(x, \mathrm{d}y) \int_{B-y} p^{\alpha_1}_{t_1}(y, \mathrm{d}z) \\ &\le ct_1^{-d/\alpha_1} = ct^{-d/\alpha} \end{aligned}$$

(where $p_{t-t_1}(x, \mathrm{d}y)$ stands for $p^{(\alpha_2)}_{t_2} * \cdots * p^{(\alpha_K)}_{t_K}(x, \mathrm{d}y)$), and we may continue writing the long equality as

$$\le ct^{-d/\alpha} \sum_{j=1}^{K} \mathbf{E} N^{(j)}_t(\mathbb{R}^d) \le ct^{-d/\alpha}.$$

Summarizing we obtain

$$\mathbf{P}^{\theta}_{x,i}\left\{N_t(B \times \mathbf{K}) > 0\right\} \le c\, t^{-d/\alpha} + c\, t^{1-\eta}.$$

■

Besides Lemma 11, our other key tool is an analogue of Lemma 3 in [15]. Recall the notations after (4.14). The proof is an easy multidimensional extension of the proof in [15].

Lemma 12 *If $\eta - 1 > d/\alpha$, then for any $x \in \mathbb{R}^d, t > 0, i \in \mathbf{K}$ and $u \in (0, t - c_2^{\alpha/d})$,*

$$\mathbf{P}_{x,i}\left\{N_t(B \times \mathbf{K}) > 0\right\} \leq q^{(i)}\left(u; 1 - c_2\,(t-u)^{-d/\alpha}\right),$$

where the constant c_2 is given in Lemma 11.

Proof. Let $|N_r| \equiv (N_r^1(\mathbb{R}^d), \ldots, N_r^K(\mathbb{R}^d))$, $r \geq 0$. For any $u \in (0, t)$,

$$\begin{aligned}\mathbf{P}_{x,i}\left\{N_t(B \times \mathbf{K}) > 0\right\} &= \sum_{\mathbf{k} \neq (0,\ldots,0)} \mathbf{P}_{x,i}\left\{|N_u| = \mathbf{k}, N_t(B \times \mathbf{K}) > 0\right\} \\ &= \mathbf{P}_{x,i}\left\{|N_u| \neq 0\right\} - \sum_{\mathbf{k} \neq (0,\ldots,0)} \mathbf{P}_{x,i}\left\{|N_u| = \mathbf{k}, N_t(B \times \mathbf{K}) = 0\right\},\end{aligned} \tag{4.20}$$

where

$$\mathbf{P}_{x,i}\left\{|N_u| = \mathbf{k}, N_t(B \times \mathbf{K}) = 0\right\} = \mathbf{E}\left[\mathbf{P}\left\{N_t(B \times \mathbf{K}) = 0 \big| |N_u| = \mathbf{k}, \Theta_{\mathbf{k}}, Y_{\mathbf{k}}\right\} I(|N_u| = \mathbf{k})\right].$$

Here $\Theta_{\mathbf{k}}$ is the vector of ages, and $Y_{\mathbf{k}}$ the vector of positions of individuals alive at time u. Using independence and Lemma 11, the conditional probability inside the above expectation gives

$$\begin{aligned}&\mathbf{P}\left\{N_t(B \times \mathbf{K}) = 0 \big| |N_u| = \mathbf{k}, \Theta_{\mathbf{k}}, Y_{\mathbf{k}}\right\} \\ &\qquad = \prod_{j=1}^{K} \prod_{l=1}^{k_j} \mathbf{P}_{y_{l,j},j}^{\theta_{l,j}}\left\{N_{t-u}(B \times \mathbf{K}) = 0\right\} \\ &\qquad = \prod_{j=1}^{K} \prod_{l=1}^{k_j} \left(1 - \mathbf{P}_{y_{l,j},j}^{\theta_{l,j}}\left\{N_{t-u}(B \times \mathbf{K}) > 0\right\}\right) \\ &\qquad \geq \left(1 - c_2\,(t-u)^{-d/\alpha}\right)^{|\mathbf{k}|},\end{aligned}$$

where in the last inequality we used that $1 - c_2\,(t-u)^{-d/\alpha} > 0$. Therefore we obtain

$$\mathbf{P}_{x,i}\left\{|N_u| = \mathbf{k}, N_t(B \times \mathbf{K}) = 0\right\} \geq \left(1 - c_2\,(t-u)^{-d/\alpha}\right)^{|\mathbf{k}|} \mathbf{P}_i\left\{|N_u| = \mathbf{k}\right\}.$$

Substituting this estimate back into (4.20), we finally get

$$\begin{aligned}\mathbf{P}_{x,i}\left\{N_t(B \times \mathbf{K}) > 0\right\} &\leq \sum_{\mathbf{k} \in \mathbb{N}^d \setminus 0} \left[1 - \left(1 - c_2\,(t-u)^{-d/\alpha}\right)^{|\mathbf{k}|}\right] \mathbf{P}_i\left\{|N_u| = \mathbf{k}\right\} \\ &= Q^{(i)}(u; 1 - c_2\,(t-u)^{-d/\alpha}, \ldots, 1 - c_2\,(t-u)^{-d/\alpha}) \\ &= q^{(i)}(u; 1 - c_2\,(t-u)^{-d/\alpha}).\end{aligned}$$

■

Let us define the set

$$\Lambda = \{\mathbf{s} \in [0,1]^K : \mathbf{f}(\mathbf{s}) \geq \mathbf{s}\}, \tag{4.21}$$

where an inequality of the form $(x_1, \ldots, x_K) \geq (y_1, \ldots, y_K)$ means here that $x_i \geq y_i$ for $i = 1, 2, \ldots, K$.

We remark that, since $\mathbf{1} - \mathbf{f}(\mathbf{1} - \mathbf{u}x) \leq M\mathbf{u}x = \mathbf{u}x$, we have $\mathbf{1} - \mathbf{u}x \in \Lambda$ for all x with $0 < \mathbf{u}x \leq \mathbf{1}$. In our case $\mathbf{u} = K^{-1}\mathbf{1}$, and this implies that the diagonal $\{(s, \ldots, s) : s \in [0, 1]\}$ is contained in Λ.

For given matrix families $A(t) = (a_{ij}(t))_{i,j}$ and $B(t) = (b_{ij}(t))_{i,j}$, $t \geq 0$, let us define the matrix convolution $C = A * B$ by

$$c_{ij}(t) = \sum_{k=1}^{K} \int_0^t a_{ik}(t-s) b_{kj}(\mathrm{d}s).$$

The convolution of a matrix and a vector is defined analogously. Put $M_\Gamma^1(t) = (m_{ij}\Gamma_i(t))_{i,j}$ and recursively define

$$M_\Gamma^{n+1}(t) = M_\Gamma^1(t) * M_\Gamma^n(t), \quad n = 1, 2, \ldots.$$

Put also $M_\Gamma^0(t) = (\delta_{ij}\Gamma_i^0(t))_{i,j}$, where $\Gamma_i^0(t)$ is the distribution function of a constant 0 random variable. Notice that $M_\Gamma^0(t)$ constitutes the unit element in matrix convolution. The following multidimensional comparison lemma is borrowed from [13], which is a generalisation of Goldstein's comparison lemma [5].

Lemma 13 *For any $t > 0$, any natural n and for all $\mathbf{s} \in \Lambda$,*

$$\begin{aligned} \mathbf{1} - \mathbf{f}_n(\mathbf{s}) - M_\Gamma^n * [(\mathbf{1} - \mathbf{s}) \otimes \Gamma](t) &\leq \mathbf{1} - F(t; \mathbf{s}) \\ &\leq \mathbf{1} - \mathbf{f}_n(\mathbf{s}) + \sum_{j=0}^{n-1} M_\Gamma^j * [(\mathbf{1} - \mathbf{s}) \otimes [\mathbf{1} - \Gamma]](t). \end{aligned}$$

Here $\mathbf{x} \otimes \mathbf{y} := (x_1y_1, x_2y_2, \ldots, x_Ky_K)$ if $\mathbf{x} = (x_1, \ldots, x_K)$ and $\mathbf{y} = (y_1, \ldots, y_K)$.

We are going to use below the upper bound given in Lemma 13. The following lemma is Lemma 5 in [14].

Lemma 14 *Consider two critical multitype branching processes sharing the same branching mechanism, with corresponding lifetime distributions $\Gamma(t) = (\Gamma_1(t), \ldots, \Gamma_K(t))$ and $\Gamma^*(t) = (\Gamma_1^*(t), \ldots, \Gamma_K^*(t))$. Assume that $\Gamma(t) \geq \Gamma^*(t)$ for all $t \geq 0$. Then for all $t \geq 0$ and $\mathbf{s} \in \Lambda$,*

$$F(t; \mathbf{s}) \leq F^*(t; \mathbf{s}),$$

where F and F^* are, respectively, the vector generating functions of the number of particles at time t in the first and second process.

The main result in this section is the following theorem.

Theorem 2 *Assume that* (4.16) *holds, the mean matrix M is stochastic, and the lifetimes satisfy $1 - \Gamma_1(t) \sim t^{-\gamma}$ for some constant $\gamma \leq 1$, and*

$$1 - \Gamma_j(x) \leq A\, x^{-\eta_j}, \quad j = 2, 3, \ldots, K,$$

where $\eta_j > 1$, $j = 2, 3, \ldots, K$. Put $\eta = \min\{\eta_j : j = 2, 3, \ldots, K\}$. If $\eta - 1 > d/\alpha$ and $d < \frac{\alpha\gamma}{\beta}$, then the process suffers local extinction.

Proof. Define the distribution function

$$\tilde{\Gamma}(t) = \prod_{i=1}^{K} \Gamma_i(t),$$

which is the distribution function of $\tilde{\xi} = \max\{\xi_1, \ldots, \xi_K\}$, where the random variables ξ_i, $i = 1, \ldots, K$, are independent with distribution function Γ_i. Lemma 15 below shows that $1 - \tilde{\Gamma}(t) \sim t^{-\gamma}$. Consider a new branching process where the branching mechanism is unchanged, but the lifetimes of all types have distribution $\tilde{\Gamma}$, and let $\tilde{F}(t; \mathbf{s})$ denote its generating function at time t. Clearly, the choice of $\tilde{\Gamma}$ shows that Lemma 14 is applicable, and so for $\mathbf{s} \in \Lambda$,

$$\tilde{F}(t; \mathbf{s}) \le F(t; \mathbf{s}). \tag{4.22}$$

(Notice that Λ, as defined in (4.21), depends only on the branching mechanism of our process). Now we apply the comparison lemma for this new process. Since now all the lifetimes have the same distribution,

$$M_{\tilde{\Gamma}}^{n}(t) = M^n \, \tilde{\Gamma}^{*n}(t),$$

where ^{*n} stands for the n-fold convolution. Moreover, for $\mathbf{s} = s\,\mathbf{1}$,

$$M_{\tilde{\Gamma}}^{j} * [(\mathbf{1} - \mathbf{s})(\mathbf{1} - \tilde{\Gamma})](t) = (1 - s)(\tilde{\Gamma}^{*j}(t) - \tilde{\Gamma}^{*(j+1)}(t)) M^j \mathbf{1} = (1 - s)(\tilde{\Gamma}^{*j}(t) - \tilde{\Gamma}^{*(j+1)}(t))\mathbf{1},$$

where we used the simple fact that M^j is stochastic if M is stochastic. Thus, in the rightmost inequality of Lemma 13 we get a telescopic sum, and therefore we obtain

$$\mathbf{1} - \tilde{F}(t; \mathbf{1}s) = \tilde{Q}(t; \mathbf{1}s) \le \mathbf{1} - \mathbf{f}_n(\mathbf{1}s) + (1 - s)[1 - \tilde{\Gamma}^{*n}(t)]\mathbf{1}.$$

According to (4.16), for the survival probabilities we have

$$1 - f_n^{(i)}(\mathbf{1}s) \le 1 - f_n^{(i)}(0) \le c\, n^{-\frac{1}{\beta}}.$$

Taking into account (4.22) we have, for $s \in (0, 1)$,

$$Q^{(i)}(t; \mathbf{1}s) = 1 - F^{(i)}(t; \mathbf{1}s) \le c\, n^{-\frac{1}{\beta}} + (1 - s)\mathbf{P}\left\{S_n > t\right\}.$$

Choosing $n = t^{\gamma/(1+\varepsilon)}$ and using that, by Lemma 5,

$$\mathbf{P}\left\{S_n > t\right\} = \mathbf{P}\left\{S_n > n^{\frac{1+\varepsilon}{\gamma}}\right\} \le 2\, n^{-\varepsilon} = 2\, t^{-\gamma\varepsilon/(1+\varepsilon)},$$

we get

$$q^{(i)}(t; 1 - s) = Q^{(i)}(t; (1 - s)\mathbf{1}) \le c\, t^{-\frac{\gamma}{(1+\varepsilon)\beta}} + s\, t^{-\gamma\varepsilon/(1+\varepsilon)}.$$

Hence, choosing $u = t/2$ in Lemma 12 we obtain the inequality

$$q^{(i)}(u; 1 - c_2\,(t - u)^{-d/\alpha}) \le c\, t^{-\frac{\gamma}{(1+\varepsilon)\beta}} + c\, t^{-\frac{d}{\alpha}} t^{-\frac{\gamma\,\varepsilon}{1+\varepsilon}}. \tag{4.23}$$

Multiplying by $t^{d/\alpha}$, the second term in the right of (4.23) goes to 0, while in the first one the exponent of t becomes

$$\frac{d}{\alpha} - \frac{\gamma}{(1+\varepsilon)\beta},$$

and this is negative if $d < \alpha\gamma/\beta$ and ε is small enough. ∎

The simple lemma we used above is the following:

Lemma 15 *Let X, Y be independent non-negative random variables with corresponding distribution functions F and G. Assume that $1 - F(x) \sim x^{-\gamma}$ and $\mathbf{E}Y < \infty$. Then for the distribution of $Z = \max\{X, Y\}$ we have*

$$1 - H(z) := \mathbf{P}\{Z > z\} \sim z^{-\gamma},$$

as $z \to \infty$.

Proof. Since $\mathbf{E}Y < \infty$, we have $y(1 - G(y)) \to 0$. Hence,

$$\begin{aligned} z^\gamma[1 - H(z)] &= z^\gamma [1 - \mathbf{P}\{\max\{X,Y\} \le z\}] \\ &= z^\gamma [1 - F(z)G(z)] = z^\gamma [1 - F(z) + F(z)(1 - G(z))] \\ &= z^\gamma [1 - F(z)] + z^\gamma F(z) [1 - G(z)] \to 1. \end{aligned}$$

∎

4.3 A lifetime with infinite mean – Case B

Now let us investigate the case when α_1 is not the minimal $\alpha = \min\{\alpha_i : i = 1, 2, \ldots, K\}$. Without loss of generality, let us assume that $\alpha = \alpha_2$.

Notice that Lemma 11 is true in this case with exponent $-d/\alpha_1$, and so the variation of Lemma 12 also remains true. We state it for the easier reference.

Lemma 16 *If $\eta - 1 > d/\alpha_1$, then for any $x \in \mathbb{R}^d, t > 0, i \in \mathbf{K}$ and $u \in (0, t - c_2^{\alpha_1/d})$,*

$$\mathbf{P}_{x,i}\{N_t(B \times \mathbf{K}) > 0\} \le q^{(i)}\left(u; 1 - c_2 (t-u)^{-d/\alpha_1}\right),$$

where the constant c_2 is given in Lemma 11.

Put

$$v = \max\left\{\frac{1}{\alpha_1}, \frac{\gamma}{\alpha}\right\}. \tag{4.24}$$

Lemma 17 *Assume that $\gamma\eta > d/\alpha + 1$. If $\gamma < 1$, then for any $\varepsilon > 0$, for any $i \in \{1, 2, \ldots, K\}$ and for any bounded Borel set B,*

$$\lim_{t\to\infty} \int_{|x| \ge t^{v+\varepsilon}} \mathbf{P}_{x,i}\{N_t(B \times \mathbf{K}) > 0\}\,\mathrm{d}x = 0.$$

For $\gamma = 1$ (then necessarily $v = 1/\alpha$),

$$\lim_{L\to\infty} \limsup_{t\to\infty} \int_{|x| \ge Lt^v} \mathbf{P}_{x,i}\{N_t(B \times \mathbf{K}) > 0\}\,\mathrm{d}x = 0.$$

Proof. Without loss of generality, we will assume that B is a ball with radius r centered at the origin. First, consider the case $\gamma < 1$. Put $C(t) = \{|x| \le t^{v+\varepsilon}\}$ and let $\varepsilon' < \alpha\varepsilon$. Recall the definition of $\sigma_j(t)$ after (2.1) and put

$$A = \{\sigma_2(t) \le t^{\gamma+\varepsilon'}, \sigma_3(t) \le t^{\gamma+\varepsilon'}, \ldots, \sigma_K(t) \le t^{\gamma+\varepsilon'}\},$$

namely A is the set where, for all ancestry lines, the spent time in type j up to t is less than $t^{\gamma+\varepsilon'}$ for all $j = 2, 3, \ldots, K$.

First we work on the set A^c. By Lemma 8,

$$\mathbf{P}\{A^c\} \le \sum_{j=2}^{K} \mathbf{P}\left\{\sigma_j(t) > t^{\gamma+\varepsilon'}\right\} \le K\, t^{1-\eta\gamma}.$$

According to Lemma 10,

$$\sup_{t\ge 1} \int_{|x|\ge L\, t^{1/\alpha}} \mathbf{E}_{x,i}\left[I(A^c) N_t(B \times \mathbf{K})\right] \mathrm{d}x \to 0 \quad \text{as } L \to \infty,$$

hence, it suffices to integrate on the region $t^{v+\varepsilon} \le |x| \le Lt^{1/\alpha}$. On the other hand

$$\mathbf{P}_{x,i}\left\{A^c, N_t(B \times \mathbf{K}) > 0\right\} \le \mathbf{P}\{A^c\},$$

and so

$$\int_{Lt^{1/\alpha} \ge |x| \ge t^{v+\varepsilon}} \mathbf{P}_{x,i}\left\{A^c, N_t(B \times \mathbf{K}) > 0\right\} \mathrm{d}x \le c\, t^{d/\alpha} t^{1-\eta\gamma} \to 0$$

due to our assumption.

From now on we work on A. Translation invariance of the motion shows that

$$\int_{\mathbb{R}^d \setminus C(t)} \mathbf{E}_{x,i} I(A) N_t(B \times \mathbf{K})\, \mathrm{d}x = \int_{\mathbb{R}^d \setminus C(t)} \mathbf{E}_{0,i} I(A) N_t((B-x) \times \mathbf{K})\, \mathrm{d}x.$$

By conditioning on the reduced tree, we can write

$$\mathbf{E}_{0,i} I(A) N_t((B-x) \times \mathbf{K}) = \mathbf{E}_{0,i} I(N_t \neq 0) I(A) \sum_{j=1}^{K} \sum_{m=1}^{N_t^j(\mathbb{R}^d)} \mathbf{P}_{0,i}\left\{W_j^m(t) \in B - x | \mathcal{T}_t^r\right\},$$

where $W_j^m(t) \stackrel{\mathcal{D}}{=} W(t_1, \alpha_1) + \cdots + W(t_K, \alpha_K)$ as in (4.19). Integrating we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d \setminus C(t)} \mathbf{E}_{0,i} I(A) N_t((B-x) \times \mathbf{K})\, \mathrm{d}x \\
&= \mathbf{E}_{0,i} I(N_t \neq 0) I(A) \sum_{j=1}^{K} \sum_{m=1}^{N_t^j(\mathbb{R}^d)} \int_{\mathbb{R}^d \setminus C(t)} \mathrm{d}x \int_{B-x} \mathbf{P}_{0,i}\left\{W_j^m(t) \in \mathrm{d}y | \mathcal{T}_t^r\right\}.
\end{aligned}$$

Since $|x+y| \le r$ and $|x| \ge t^{v+\varepsilon}$, we have $|y| > t^{v+\varepsilon} - r \ge t^{v+\varepsilon}/2$ for all t large enough. Using Fubini's theorem and that $\int_{|x+y|\le r} \mathrm{d}x =: c(r)$ independently of y, the double integral can be bounded from above by

$$c(r)\mathbf{P}_{0,i}\left\{ |W_j^m(t)| \ge \frac{t^{v+\varepsilon}}{2} \Big| \mathcal{T}_t^r \right\}.$$

On the event A we can write

$$\begin{aligned}
\mathbf{P}_{0,i}\left\{ |W_j^m(t)| \ge \frac{t^{v+\varepsilon}}{2} \Big| \mathcal{T}_t^r \right\} &\le \sum_{k=1}^{K} \mathbf{P}\left\{ |W(t_k,\alpha_k)| \ge \frac{t^{v+\varepsilon}}{2K} \Big| \mathcal{T}_t^r \right\} \\
&\le \sum_{k=1}^{K} \mathbf{P}\left\{ |W(t_k,\alpha_k)| \ge \frac{t^{\delta}\, t_k^{1/\alpha_k}}{2K} \Big| \mathcal{T}_t^r \right\} \\
&= \sum_{k=1}^{K} \mathbf{P}\left\{ |W(1,\alpha_k)| \ge \frac{t^{\delta}}{2K} \right\},
\end{aligned}$$

where we used that $t^{v+\varepsilon} \ge t^{\delta}\, t_1^{1/\alpha_1}$ for some small enough $\delta > 0$, and that, by the definition of A and ε', the inequalities $t^{v+\varepsilon} \ge t^{\delta} t_j^{1/\alpha} \ge t^{\delta} t_j^{1/\alpha_j}$ hold, while in the last step the self-similarity of the stable process was used. The last upper bound above goes to 0 as $t \to \infty$, and

$$\sup_{t>0} \mathbf{E}_i \sum_{j=1}^{K} N_t^j(\mathbb{R}^d) < \infty$$

due to criticality of the branching. This finishes the proof of the lemma under the assumption that $\gamma < 1$. The proof for the case $\gamma = 1$ is a straightforward adaptation of the previous one. ∎

The value $\alpha_1\gamma$ can be considered as the *effective mobility* of the type-1 particles. At an intuitive level if $\alpha_1\gamma > \alpha$, then second particle type is more mobile, even considering the long-living effect of the first one, so that in this case the "dominant" mobility is associated to the second particle type. The next two theorems deal with the cases when the first type is the dominant and when the second one, respectively.

Theorem 3 *Assume that* (4.16) *holds and that* $\gamma\eta > d/\alpha + 1$. *If* $\alpha \ge \alpha_1\gamma$, *i.e. the mobility of the first particle type is dominant, then the process suffers local extinction for* $d < \alpha_1\gamma/\beta$.

Proof. Writing $u = t/2$ in Lemma 12, and proceeding in the same way as we did to obtain (4.23) in the proof of Theorem 2, we get

$$q^{(i)}(t/2; 1 - c\,t^{-d/\alpha_1}) \le c\,t^{-d/\alpha_1} t^{-\gamma\varepsilon/(1+\varepsilon)} + c\,t^{-\gamma/(1+\varepsilon)\beta}.$$

Since in this case $v = 1/\alpha_1$, from Lemma 17 we get extinction provided that

$$\frac{d}{\alpha_1} < \frac{\gamma}{(1+\varepsilon)\beta},$$

which holds for ε small enough if $d < \alpha_1\gamma/\beta$. ∎

Theorem 4 *Assume that $\gamma\eta > d/\alpha + 1$. If $\alpha_1\gamma > \alpha$, i.e. the mobility of the second particle type is the dominant one, then the process suffers local extinction for $d < d_+$, where*

$$d_+ = \frac{\gamma}{\frac{(\beta+1)\gamma}{\alpha} - \frac{1}{\alpha_1}}. \tag{4.25}$$

Proof. From the comparison lemma (Lemma 13) we have

$$Q^{(i)}(t; \mathbf{1}s) \le c\, n^{-\frac{1}{\beta}} + (1-s)\mathbf{P}\left\{S_n \ge t\right\}.$$

We have to choose $t = n^{\frac{1+\varepsilon}{\gamma}}$ for some $\varepsilon > 0$, and then minimize the estimations in ε. In this case

$$q^{(i)}(t; 1-s) \le c\, t^{-\frac{\gamma}{(1+\varepsilon)\beta}} + s\, t^{-\frac{\varepsilon\gamma}{1+\varepsilon}}.$$

Putting $u = t/2$ in Lemma 16 renders

$$q^{(i)}(t/2; 1 - c_2\, t^{-d/\alpha_1}) \le c\, t^{-\frac{\gamma}{(1+\varepsilon)\beta}} + c\, t^{-d/\alpha_1 - \frac{\varepsilon\gamma}{1+\varepsilon}}.$$

Therefore we have to maximize

$$\min\left\{\frac{\gamma}{(1+\varepsilon)\beta}, \frac{d}{\alpha_1} + \frac{\varepsilon\gamma}{1+\varepsilon}\right\}$$

with respect to ε. Since the term $\gamma/((1+\varepsilon)\beta)$ is monotone decreasing, and the term $d/\alpha_1 + \varepsilon\gamma/(1+\varepsilon)$ is increasing in ε, easy computations show that the optimal choice is

$$\varepsilon = \frac{\gamma(1+\beta^{-1})}{d/\alpha_1 + \gamma} - 1,$$

and the estimation is

$$q^{(i)}(t/2; 1 - c_2\, t^{-d/\alpha_1}) \le c\, t^{-\frac{d/\alpha_1+\gamma}{1+\beta}}.$$

Combining this with Lemma 17, and taking into account that $v = \gamma/\alpha$, we get extinction if

$$d\frac{\gamma}{\alpha} < \frac{d/\alpha_1 + \gamma}{1+\beta}.$$

Solving the inequality, gives that extinction holds for $d < d_+$, with the anticipated dimension d_+. ∎

Remark Notice that if $\gamma/\alpha - 1/\alpha_1 \to 0$, that is, if the effective mobilities of types 1 and 2 are approximately the same, then $d_+ \to \alpha_1\gamma/\beta$, which is the critical dimension in Theorem 3. Moreover, for fixed α, α_1 and γ, the critical dimension d_+ considered as a function of β, is decreasing, which is consistent with the known results.

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